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On the density of states of Schrödinger operators with a random potential

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Abstract. Using very recent results on ergodic theorems for superadditive processes on $\mathbb{R}^d$, we prove the existence of the density of states for a wide class of random Schrödinger operators. In particular, new non-asymptotic estimates on the density of states are obtained and examples are discussed.

1. Introduction

Schrödinger operators $H_\omega = -\Delta + V_\omega$ with a random potential V_ω occur naturally in models of disordered systems (see e.g. Lifschitz 1965); a quantity of particular interest is the 'integrated' density of states $\rho(\lambda)$ defined as the thermodynamic limit

$$\rho(\lambda) = \lim_{\Lambda \nearrow \mathbb{R}^d} \frac{1}{|\Lambda|} \rho_\Lambda(\lambda, \omega)$$

where $\rho_\Lambda(\lambda, \omega)$ is the number of eigenvalues less than λ of H_ω restricted to a finite box Λ with appropriate boundary conditions. A number of papers in both mathematical and physical literature have been devoted to the proof of the existence of the above limit and its independence of the particular realisation of V_ω, and to the estimate of its asymptotic behaviour as λ goes to either the left edge of the spectrum of H_ω or to plus infinity. We refer the reader to the work by Pastur (1973, 1971, 1972, 1977), Fukushima (1974, 1980), Fukushima and Nakao (1977), Nakao (1977) and Kotani (1976). The main tool for their proofs is the representation of the Laplace transform of $\rho_\Lambda(\lambda, \omega)$ as a Wiener integral, together with estimates on the asymptotic behaviour of such integrals for large times. However, a key assumption was the strict stationarity and ergodicity of the potential V_ω as well as some regularity of its sample paths. Therefore the method does not cover the following physically interesting case when V_ω is a random modification of a periodic structure:

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} f_i(\omega, x - x_i) \tag{1}$$

where $\{x_i\}_{i \in \mathbb{Z}^d}$ is a lattice in $\mathbb{R}^d$ and $f_i(\omega)$ is a random function. An example of such a potential is the case when $f_i(\omega, x) = q_i(\omega) f[x - \xi_i(\omega)]$ where $\{q_i(\omega)\}_{i \in \mathbb{Z}^d}$ and $\{\xi_i(\omega)\}_{i \in \mathbb{Z}^d}$ are two ergodic random fields on $\mathbb{Z}^d$, which models a solid with 'particles' with random

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charges $q_i(\omega)$ located at the random positions $x_i + \xi_i(\omega)$. Hamiltonians of such a kind have been studied in Kirsch (1981) and Kirsch and Martinelli (1981a, b, c, 1980) (see also Lieb and Mattis (1966) for physical background and specific models). In this paper we present a new proof for the existence and non-randomness of the density of states, as well as new non-asymptotic estimates, which, besides generalising previous results for metrically transitive potentials, also covers case (1). Our results are based on well known estimates on the number of bound states of Schrödinger operators (see Lieb and Thirring (1976) and Lieb (1976)) and on very recent results on ergodic theorems for superadditive processes due to Akcoglu and Krengel (1981). We also refer the reader to the paper by Slivnyak (1966) for a somewhat similar approach to the problem. We now briefly present the contents of the various sections. In § 2 we recall and prove some simple results concerning Schrödinger operators in a finite box Λ. In § 3 we prove the existence of $\rho(\lambda)$ as well as the basic estimates, and we discuss the dependence of $\rho(\lambda)$ on the boundary conditions. In § 4 we prove some asymptotic results for $\lambda \to \pm\infty$ which are then applied first to the case when V_ω is a random step function on $\mathbb{R}^d$ and then to the case when V_ω is an arbitrary gaussian random field. These examples illustrate the difference between the density of states computed in the continuous and discrete cases (tight-binding model) when the Laplacian is replaced by its discrete version on $l^2(\mathbb{Z}^d)$. Other interesting questions concerning $\rho(\lambda)$, such as its support properties as a measure on the real line, the computations of the Lifshitz exponents (see Lifshitz (1965) and Romerio and Wreszinski (1979)) for some cases, as well as the extension of the present results to the case of N interacting particles subjected to an external random field, will be discussed in a forthcoming paper.

2. Schrödinger operators in finite boxes

In this section we state and prove some results concerning Schrödinger operators in a finite box $\Lambda \subset \mathbb{R}^d$, $d \geq 1$, which will be needed later when we will discuss random Schrödinger operators. In the sequel Λ will denote an arbitrary bounded open hypercube in $\mathbb{R}^d$.

Definition 2.1. Let $H^1(\Lambda) = \{f \in L^2(\Lambda); \nabla f \in L^2(\Lambda)\}$ where the gradient is intended in the distributional sense. It is well known (see e.g. Adams 1975) that $H^1(\Lambda)$ is a Hilbert space under the norm $\|f\|_{H^1(\Lambda)} \equiv (\Sigma_{|\alpha| \leq 1} \|D^\alpha f\|_2^2)^{1/2}$ where $\|\cdot\|_2$ is the $L^2(\Lambda)$ norm and D^α are the usual distributional derivatives. Let also $H_0^1(\Lambda)$ be the closure of $C_0^\infty(\Lambda)$ in the norm $\|\cdot\|_{H^1(\Lambda)}$. We then define (see e.g. Reed and Simon 1978b) the *Dirichlet Laplacian* in Λ, $-\Delta_\Lambda^{\mathrm{D}}$, as the unique self-adjoint operator on $L^2(\Lambda)$ whose quadratic form is the closure of the form

$$q^{\mathrm{D}}(f, g) = \int_\Lambda (\overline{\nabla f} \cdot \nabla g)(x)\, \mathrm{d}x, \qquad f, g \in C_0^\infty(\Lambda).$$

The *Neumann Laplacian* in Λ, $-\Delta_\Lambda^{\mathrm{N}}$, is the unique self-adjoint operator on $L^2(\Lambda)$ whose quadratic form is

$$q^{\mathrm{N}}(f, g) = \int_\Lambda (\overline{\nabla f} \cdot \nabla g)(x)\, \mathrm{d}x, \qquad f, g \in H^1(\Lambda).$$

The following theorem is well known (see e.g. Reed and Simon 1978b).

Theorem 2.1. Both $-\Delta_\Lambda^{\mathrm{D}}$ and $-\Delta_\Lambda^{\mathrm{N}}$ have compact resolvent, so that their spectra consist of isolated eigenvalues of finite multiplicity, denoted by μ_k^{D} and μ_k^{N} respectively, such that $\mu_k^{\mathrm{D}} \to +\infty$, $\mu_k^{\mathrm{N}} \to +\infty$ as $k \to +\infty$.

In order to define the Schrödinger operators $-\Delta_\Lambda^{\mathrm{D}} + V$, $-\Delta_\Lambda^{\mathrm{N}} + V$ in Λ we need the following lemma.

Lemma 2.1. Let $V \in L^p(\Lambda)$ with $p = 1$ if $d = 1$, $p > 1$ if $d = 2$, and $p \geq \frac{1}{2}d$ if $d \geq 3$. Then V as an operator on $L^2(\Lambda)$ is a small form perturbation of both $-\Delta_\Lambda^{\mathrm{D}}$ and $-\Delta_\Lambda^{\mathrm{N}}$.

Proof. We have to show that $\forall f \in H^1(\Lambda)$

$$\left| \int_\Lambda V(x) f(x)^2 \,\mathrm{d}x \right| \leq \varepsilon \int_\Lambda |\nabla f(x)|^2 \,\mathrm{d}x + b(\varepsilon, V) \|f\|_2^2 \tag{2}$$

$\forall \varepsilon > 0$ and some constant $b(\varepsilon, V) \geq 0$. Since V can be split into $V = V_\infty + W$ where $V_\infty \in L^\infty(\Lambda)$ and $W \in L^p(\Lambda)$, and furthermore for any $\varepsilon > 0$ V_∞ can be chosen so large that $\|W\|_p < \varepsilon$, (2) follows immediately from the usual Sobolev embedding theorems (see e.g. Adams 1975 and Faris 1975).

Remark 2.1. It is worthwhile to note that the constant $b(\varepsilon, V)$ can be chosen as small as we like for $\|V\|_p$ sufficiently small.

Let us now denote by H_Λ^{D} and H_Λ^{N} the operators $-\Delta_\Lambda^{\mathrm{D}} + V$, $-\Delta_\Lambda^{\mathrm{N}} + V$ respectively, defined for V as in lemma 2.1 as form sums (see e.g. Reed and Simon 1978a). For both H_Λ^{D} and H_Λ^{N} one has the following perturbation result.

Proposition 2.1.

(a) Both H_Λ^{D} and H_Λ^{N} have compact resolvent.

(b) If we denote by $\lambda_n(H_\Lambda^{\mathrm{D}})$, $\lambda_n(H_\Lambda^{\mathrm{N}})$ the eigenvalues (counting multiplicity) of H^{D}, H^{N} respectively, then $\forall \varepsilon > 0$

$$-b(\varepsilon, V) + (1-\varepsilon)\mu_n^{\mathrm{D}} \leq \lambda_n(H_\Lambda^{\mathrm{D}}) \leq (1+\varepsilon)\mu_n^{\mathrm{D}} + b(\varepsilon, V),$$

$$-b(\varepsilon, V) + (1-\varepsilon)\mu_n^{\mathrm{N}} \leq \lambda_n(H_\Lambda^{\mathrm{N}}) \leq (1+\varepsilon)\mu_n^{\mathrm{N}} + b(\varepsilon, V),$$

where μ_n^{D}, μ_n^{N} are the eigenvalues of $-\Delta_\Lambda^{\mathrm{D}}$ and $-\Delta_\Lambda^{\mathrm{N}}$ respectively.

(c) $\exp(-tH_\Lambda^{\mathrm{D}})$, $\exp(-tH_\Lambda^{\mathrm{N}})$, $t > 0$, are trace class.

(d) If $V_n \in L^p(\Lambda)$, $V \in L^p(\Lambda)$ and $\|V - V_n\|_p \to 0$ as $n \to +\infty$, where p is as in lemma 2.1, then

$$\lambda_k(H_n^{\mathrm{D}}) \to \lambda_k(H_\Lambda^{\mathrm{D}}), \qquad \lambda_k(H_n^{\mathrm{N}}) \to \lambda_k(H_\Lambda^{\mathrm{N}}) \qquad \text{as } n \to +\infty.$$

(e) $\mathrm{Tr}[\exp(-tH_n^{\mathrm{D}})] \to \mathrm{Tr}[\exp(-tH_\Lambda^{\mathrm{D}})]$ as $n \to +\infty$, $\forall t > 0$, and the same holds for H_n^{N}, H_Λ^{N}. Here Tr denotes the trace.

Proof.

(a) It follows from theorem XIII.68 of Reed and Simon (1978b).

(b) It is a straightforward consequence of the min–max principle and lemma 2.1.

(c) It follows from (b) and the fact that both $\exp(t\Delta_\Lambda^{\mathrm{D}})$ and $\exp(t\Delta_\Lambda^{\mathrm{N}})$, $t > 0$, are trace class.

(d) From the min-max principle and lemma 2.1 applied to $V-V_n$ we have that for any $\varepsilon>0$

$$-b(\varepsilon, V-V_n)+\lambda_k[-(1-\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V]\leqslant\lambda_k(H_n^{\mathrm{D}})\leqslant\lambda_k[-(1+\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V]+b(\varepsilon, V-V_n).$$

Since $b(\varepsilon, V-V_n)$ can be taken arbitrarily small for $n\to+\infty$, we have that if $\lambda_k^{\mathrm{D}}(\infty)$ is an accumulation point of $\lambda_k(H_n^{\mathrm{D}})$ then

$$\lambda_k[-(1-\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V]\leqslant\lambda_k^{\mathrm{D}}(\infty)\leqslant\lambda_k[-(1+\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V] \qquad \forall\varepsilon>0.$$

If we observe now that $-(1+\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V$ $(-(1-\varepsilon)\Delta_\Lambda^{\mathrm{D}}+V)$ are families of closed, semibounded quadratic forms decreasing (increasing), as $\varepsilon\to0$, to the quadratic form $-\Delta_\Lambda^{\mathrm{D}}+V$, we can apply a general result due to Robinson (1971) (see also Simon 1978 and Weidmann 1980) to get that $\lambda_k^{\mathrm{D}}(\infty)=\lambda_k(H^{\mathrm{D}})$. The same argument clearly applies to $\lambda_k(H_n^{\mathrm{N}})$.

(e) It is an immediate consequence of (d) and (c).

Remark 2.2. Actually for the definition of the operators H_Λ^{N} and H_Λ^{D} it is not necessary for the potential V to be a function. It can also be a distribution, provided it is small in the sense of quadratic forms with respect to the Dirichlet and Neumann Laplacians. This is the case, for example, when $V(x)=\Sigma_{i=1}^n\delta(x-x_i)$, $(x_i)_{i=1}^n$ being n arbitrary different points in Λ and δ the usual delta function at $x=0$.

By means of the above proposition we are now able to define, for a fixed hypercube Λ in $\mathbb{R}^d$, two positive non-decreasing functions of bounded variation in $\mathbb{R}$, $\rho_\Lambda^{\mathrm{D}}(\lambda)$, $\rho_\Lambda^{\mathrm{N}}(\lambda)$ as follows.

Definition 2.2.

$$\begin{aligned}\rho_\Lambda^{\mathrm{D}}(\lambda)&=\#\{k\in\mathbb{N}\colon \lambda_k(H^{\mathrm{D}})<\lambda\},\\ \rho_\Lambda^{\mathrm{N}}(\lambda)&=\#\{k\in\mathbb{N};\ \lambda_k(H^{\mathrm{N}})<\lambda\},\end{aligned}\qquad \lambda\in\mathbb{R}.$$

The following monotonicity properties of $\rho_\Lambda^{\mathrm{D}}(\lambda)$ and $\rho_\Lambda^{\mathrm{N}}(\lambda)$ with respect to the box Λ, known as the Neumann–Dirichlet bracketing, will be useful later on. In the sequel we assume that if $V\in L^p(\Lambda')$, p as in lemma 2.1, then $\forall\Lambda\subset\Lambda'$, $\rho_\Lambda^{\mathrm{D}}(\lambda)$, $\rho_\Lambda^{\mathrm{N}}(\lambda)$ are computed for the operators $-\Delta_\Lambda^{\mathrm{D}}+V_\Lambda$, $-\Delta_\Lambda^{\mathrm{N}}+V_\Lambda$, on $L^2(\Lambda)$ where V_Λ is the restriction of V to Λ.

Proposition 2.2.

(a) If $\Lambda\subset\Lambda'$ then $\rho_{\Lambda'}^{\mathrm{D}}(\lambda)\geqslant\rho_\Lambda^{\mathrm{D}}(\lambda)$, $\qquad \lambda\in\mathbb{R}$.

(b) $\forall\Lambda\subset\mathbb{R}^d$, $\qquad \rho_\Lambda^{\mathrm{N}}(\lambda)\geqslant\rho_\Lambda^{\mathrm{D}}(\lambda)$, $\qquad \lambda\in\mathbb{R}$.

(c) Let Λ_1, Λ_2 be disjoint open hypercubes such that $(\overline{\Lambda_1\cup\Lambda_2})^{\mathrm{int}}=\Lambda$ and $\Lambda\backslash(\Lambda_1\cup\Lambda_2)$ has measure zero. Then

$$\begin{aligned}\rho_\Lambda^{\mathrm{D}}(\lambda)&\geqslant\rho_{\Lambda_1}^{\mathrm{D}}(\lambda)+\rho_{\Lambda_2}^{\mathrm{D}}(\lambda), &&\lambda\in\mathbb{R},\\ \rho_\Lambda^{\mathrm{N}}(\lambda)&\leqslant\rho_{\Lambda_1}^{\mathrm{N}}(\lambda)+\rho_{\Lambda_2}^{\mathrm{N}}(\lambda), &&\lambda\in\mathbb{R}.\end{aligned}$$

For a proof of the above proposition we refer the reader to Reed and Simon (1978b, ch XIII, 15) (see also Glimm and Jaffe 1981).

Finally we will need the following asymptotic result (see e.g. Lieb 1976, Reed and Simon 1978b and Simon 1979).

Proposition 2.3.

$$\lim_{\lambda\to+\infty} \rho_\Lambda^{\rm D}\frac{(\lambda)}{\lambda^{d/2}}=\frac{\tau_d}{(2\pi)^d}|\Lambda|, \qquad \lim_{\lambda\to+\infty} \rho_\Lambda^{\rm N}\frac{(\lambda)}{\lambda^{d/2}}=\frac{\tau_d}{(2\pi)^d}|\Lambda|,$$

where τ_d is the volume of the unit sphere in $\mathbb{R}^d$ and $|\Lambda|$ is the Lebesgue measure of Λ.

Proof. We give the proof only for $\rho_\Lambda^{\rm D}(\lambda)$ since $\rho_\Lambda^{\rm N}(\lambda)$ can be treated in the same way. From point (b) of proposition 2.1 we have

$$\rho_\Lambda^{\rm D}(\lambda)\leq \#\left\{k\in\mathbb{N};\ \mu_k^{\rm D}<\frac{\lambda+b(\varepsilon,V)}{(1-\varepsilon)}\right\}, \qquad \forall\varepsilon>0,$$

$$\rho_\Lambda^{\rm D}(\lambda)\geq \#\left\{k\in\mathbb{N};\ \mu_k^{\rm D}<\frac{\lambda-b(\varepsilon,V)}{1+\varepsilon}\right\}, \qquad \forall\varepsilon>0.$$

The result now follows from Weyl's result on

$$F_\Lambda(\lambda)=\#\{k\in\mathbb{N};\ \mu_k^{\rm D}<\lambda\}, \qquad \lim_{\lambda\to+\infty}\frac{F_\Lambda(\lambda)}{\lambda^{d/2}}=\frac{\tau_d}{(2\pi)^d}|\Lambda|.$$

3. Existence of the density of states and independence of boundary conditions

Let $V(x,\omega)$ be a jointly measurable random field on $\mathbb{R}^d$ such that:

(a) There exists a group of measure preserving transformations $\{T_i\}_{i\in I}$, $I=\mathbb{R}^d$ or $I=\mathbb{Z}^d$, in the probability space $(\Omega,\tilde{\mathscr{F}},P)$ such that $V(x,T_i\omega)=V(x-i,\omega)$ $\forall i\in I$, and $\{T_i\}_{i\in I}$ is metrically transitive in the sense that if $T_iA=A$ $\forall i\in I$, $A\in\tilde{\mathscr{F}}$, then $P(A)$ is either one or zero.

(b) If $I=\mathbb{R}^d$ then $E\{|V(0,\omega)|^p\}<+\infty$. If $I=\mathbb{Z}^d$ then $E\{\int_{\Lambda_0}|V(x,\omega)|^p\,{\rm d}x\}<+\infty$. Here p is as in lemma 2.1, E denotes the expectation with respect to the measure P on $(\Omega,\tilde{\mathscr{F}})$ and $\Lambda_0=\{\{x_i\}_{i=1}^d\in\mathbb{R}^d;\ -\frac{1}{2}\leq x_i\leq\frac{1}{2}\}$. In the sequel we will refer to (a) and (b) as *assumption A*. We will also denote by $H_\Lambda^{\rm D}(\omega)$, $H_\Lambda^{\rm N}(\omega)$ the operators $-\Delta_\Lambda^{\rm D}+V(\cdot,\omega)$, $-\Delta_\Lambda^{\rm N}+V(\cdot,\omega)$ respectively, and by $\rho_\Lambda^{\rm D}(\lambda,\omega)$, $\rho_\Lambda^{\rm N}(\lambda,\omega)$ the corresponding distribution functions defined as in § 2. We notice that $H_\Lambda^{\rm N}(\omega)$ and $H_\Lambda^{\rm D}(\omega)$ for a fixed hypercube Λ are well defined for almost all $\omega\in\Omega$, since $V(\cdot,\omega)\in L^p(\Lambda)$ almost surely using assumption A (b), and that using the general results of Kirsch and Martinelli (1981a) both $\rho_\Lambda^{\rm D}(\lambda,\omega)$ and $\rho_\Lambda^{\rm N}(\lambda,\omega)$ are measurable in λ and ω.

Proposition 3.1. Let $\Lambda\subset\mathbb{R}^d$ be a fixed bounded hypercube and assume that the random field $V(x,\omega)$ satisfies assumption A. Then

(a) if $d\geq 3$ $$E\{\rho_\Lambda^{\rm D}(\lambda,\omega)\}\leq C_dE\left\{\int_\Lambda |(V(x,\omega)-\lambda)^-|^{d/2}\,{\rm d}x\right\}$$

for some positive constant C_d, where for $V\in L^p(\Lambda)$, $V^-=\max(V,0)$;

(b) if $d=2$ $$E\{\rho_\Lambda^{\rm D}(\lambda,\omega)\}\leq C_2E\left\{\int_\Lambda |(V(x,\omega)-\lambda-\eta)^-|^p\,{\rm d}x\right\}\Big/\eta^{p-1}$$

for some positive constant C_2, any $\eta > 0$ and any $p > 1$;

(c) if $d = 1$ $\qquad E\{\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)\} \leq C_1 E\left\{\int_\Lambda |(V(x,\omega) - \lambda - \eta)^-| \, \mathrm{d}x\right\} \Big/ \sqrt{\eta}$

for some positive constant C_1 and any $\eta > 0$.

Proof.

(a) By definition

$$\begin{aligned}\rho_\Lambda^{\mathrm{D}}(\lambda, \omega) &= \#\{k \in \mathbb{N}; \lambda_k[H_\Lambda^{\mathrm{D}}(\omega)] < \lambda\} \\ &= \#\{k \in \mathbb{N}; \lambda_k[H_\Lambda^{\mathrm{D}}(\omega) - \lambda] < 0\} \\ &\leq \#\{k \in \mathbb{N}; \lambda_k[-\Delta + (V_\omega - \lambda)^- \chi_\Lambda] < 0\}\end{aligned} \tag{3}$$

by proposition 3.1 and the min–max principle, where $\lambda_k[-\Delta + (V_\omega - \lambda)^- \chi_\Lambda]$ are the eigenvalues (given by the min–max) of $-\Delta + (V_\omega - \lambda)^- \chi_\Lambda$ in $L^2(\mathbb{R}^d)$. Here $\chi_\Lambda(x) = 1$ if $x \in \Lambda$, $\chi_\Lambda(x) = 0$ otherwise. By the Cwikel–Lieb–Rosenbljum bound (see e.g. Reed and Simon 1978b) the RHS of (3) is less than

$$C_d \int_\Lambda |(V(x,\omega) - \lambda)^-|^{d/2} \, \mathrm{d}x, \qquad C_d > 0.$$

(b) As in the proof of (a) we can bound $\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)$ by

$$\rho_\Lambda^{\mathrm{D}}(\lambda, \omega) \leq \#\{k \in \mathbb{N}; \lambda_k[-\Delta + (V_\omega - \lambda - \eta)^- \chi_\Lambda] < -\eta\} \qquad \forall \eta > 0. \tag{4}$$

Using now the Lieb–Thirring bound (see Lieb and Thirring 1976), we obtain that (4) is bounded by

$$C_2 \frac{1}{\eta^{p-1}} \int_\Lambda |(V_\omega - \lambda - \eta)^-|^p \, \mathrm{d}x, \qquad C_2 > 0, \forall p > 1.$$

(c) This is proved in exactly the same way.

In order to prove the main theorem of this section we need the following result due to Akcoglu and Krengel (1981). Let $\mathscr{F}$ be the class of sets $[a, b)$ of the form $[a, b) = \{\boldsymbol{x} \in \mathbb{R}^d, a_i \leq x_i < b_i, i = 1 \ldots d, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d\}$. The class $\mathscr{F}_1$ is defined analogously but with $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{Z}^d$. A family of sets $(\Lambda_r)_{r \in \tilde{Q}}$ ($\tilde{Q} \subset Q$, the field of rational numbers) in $\mathscr{F}$ is called regular if there exists another family (Λ'_r) in $\mathscr{F}$ such that

(i) $\Lambda_r \subset \Lambda'_r \; \forall r$;

(ii) $\Lambda'_r \subset \Lambda'_s$ whenever $r < s$;

(iii) $0 < |\Lambda'_r| \leq C|\Lambda_r| \; \forall r$ and some constant $C > 0$, where $|\Lambda_r|$ denotes the Lebesgue measure of the set Λ_r. If furthermore the family (Λ'_r) can be chosen in such a way that $\mathbb{R}^d = \bigcup_r \Lambda'_r$ then we say that $\lim_{r \to +\infty} \Lambda_r = \mathbb{R}^d$.

An analogous definition holds for a family of sets in $\mathscr{F}_1$, but in this case r ranges over the integers. Let now $T = (T_i)_{i \in I}$, I equal to either $\mathbb{R}^d$ or $\mathbb{Z}^d$, be a group of measure preserving transformations in a probability space $(\Omega, \tilde{\mathscr{F}}, P)$ as in assumption A, and let $F: \mathscr{F} \to L^1(\Omega, P)$ be a set function with the following properties in the case when $I = \mathbb{R}^d$:

(i) $F_\Lambda(T_i\omega) = F_{\Lambda + i}(\omega), \qquad \Lambda \in \mathscr{F}, \forall i \in I$;

(ii) if $\Lambda_1 \ldots \Lambda_n$ are disjoint sets in $\mathscr{F}$ and if $\Lambda = \bigcup_{i=1}^n \Lambda_i$ is also in $\mathscr{F}$ then $F_\Lambda \geq \Sigma_i^n F_{\Lambda_i}$;

(iii) $\sup\left(\frac{1}{|\Lambda|} \int F_\Lambda(\omega) \, \mathrm{d}P(\omega), \Lambda \in \mathscr{F}, |\Lambda| > 0\right) = \gamma(F) < +\infty$;

and the same when $I = \mathbb{Z}^d$ but with $\mathscr{F}$ replaced by $\mathscr{F}_1$. Then F is called a superadditive process with respect to the group $(T_i)_{i\in I}$. For such a class of processes Akcoglu and Krengel proved the following.†

Theorem 3.1. Let F be a superadditive process with respect to the group $(T_i)_{i\in\mathbb{R}^d}$ such that there exists a $\tilde{F} \in L^1(\Omega, P)$ with $F_{[a,b)} \leqslant \tilde{F}$, $\forall \boldsymbol{a}$ and $\boldsymbol{b}$ with $|a_i| \leqslant 1$, $|b_i| \leqslant 1$, a_i, $b_i \in Q$, $i = 1 \ldots d$. Let also (Λ_r) be a regular family in $\mathscr{F}$ with $\lim_{r\to+\infty} \Lambda_r = \mathbb{R}^d$. Then $\lim_{r\to+\infty} F_{\Lambda_r}(\omega)/|\Lambda_r| = \gamma(F)$ almost surely.

An analogous result holds in the discrete case $I = \mathbb{Z}^d$ for any discrete superadditive process F and any regular family (Λ_n) in $\mathscr{F}_1$. Note that in the discrete case the additional assumption $F_{[a,b)} \leqslant \tilde{F}$ is not necessary.

We are now in a position to prove the following theorem.

Theorem 3.2. Let $V(x, \omega)$ be a random field on $\mathbb{R}^d$ satisfying assumption A. Then there exists on $\mathbb{R}$ a non-decreasing positive function $\rho^{\mathrm{D}}(\lambda)$ and a set $\Omega_0 \subset \Omega$ of P-measure one such that $\forall\lambda \in Q$

$$\lim_{r\to+\infty} \frac{1}{|\Lambda_r|} \rho_{\Lambda_r}(\lambda, \omega) = \rho(\lambda) \qquad \forall\omega \in \Omega_0$$

where $\{\Lambda_r\}$ is a regular family in $\mathscr{F}$ or $\mathscr{F}_1$ increasing to $\mathbb{R}^d$, the choice of $\mathscr{F}$ or $\mathscr{F}_1$ depending on whether the set I of assumption A is $\mathbb{R}^d$ or $\mathbb{Z}^d$.

Proof. We treat only the case $I = \mathbb{R}^d$, the other one being completely analogous. We first show that for $\lambda \in Q$ fixed $\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)$, $\Lambda \in \mathscr{F}$ is a superadditive process with respect to the group $(T_i)_{i\in\mathbb{R}^d}$ in the sense explained before, and that it satisfies the additional condition of theorem 3.1. It is clear from assumption A and proposition (2.2) that the first two properties of the definition of a superadditive process are satisfied. Furthermore, from proposition 3.1, the stationarity of the random field $V(x, \omega)$ and assumption A, we have that

$$\forall\Lambda \subset \mathbb{R}^d \qquad |\Lambda|^{-1} E\{\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)\} \leqslant C_d E\{|V(0, \omega) - \lambda|^{d/2} \chi(V(0, \omega) < \lambda)\} < +\infty$$

for $d \geqslant 3$ and similarly for $d = 2$ and $d = 1$, so that $\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)$ is a superadditive process. Since $\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)$ is positive, the condition of theorem 3.1 is clearly satisfied, and thus there exists a set $\Omega_\lambda \subset \Omega$ of probability one such that

$$\lim_{r\to+\infty} \frac{1}{|\Lambda_r|} \rho_{\Lambda_r}^{\mathrm{D}}(\lambda, \omega) = \sup_{\Lambda\in\mathscr{F}} \frac{1}{|\Lambda|} E\{\rho_\Lambda^{\mathrm{D}}(\lambda, \omega)\} \equiv \rho^{\mathrm{D}}(\lambda), \qquad \forall\omega \in \Omega_\lambda.$$

Taking now $\Omega_0 = \bigcap_{\lambda\in Q} \Omega_\lambda$ and defining $\rho^{\mathrm{D}}(\lambda) = \lim_{\lambda_n\to\lambda} \rho^{\mathrm{D}}(\lambda_n)$ for $\lambda \in \mathbb{R}\backslash Q$, where λ_n is a sequence of rational numbers decreasing to λ, we obtain the statement.

An immediate corollary is the following.

Corollary 3.1. Let $\rho^{\mathrm{D}}(\lambda)$ be defined according to the previous theorem. Then
(a) if the index set I of assumption A is $\mathbb{R}^d$, for any $\lambda \in Q$:

(i) $\rho^{\mathrm{D}}(\lambda) \leqslant C_d E\{|V(0, \omega) - \lambda|^{d/2} \chi(V(0, \omega) < \lambda)\}$ if $d \geqslant 3$;

† Actually Akcoglu and Krengel worked with the semigroup (T_i), $i \in (\mathbb{R}^+)^d$ or $i \in (\mathbb{Z}^+)^d$ but, as is said in Krengel (1982), all their results extend immediately to the case we consider.

(ii) $\rho^{\mathrm{D}}(\lambda) \leqslant C_2 E\{|V(0,\omega)-\lambda-\eta|^p \chi(V(0,\omega)<\lambda+\eta)\}/\eta^{p-1}$ $\quad\forall\eta>0$

and $p>1$ such that $E\{|V(0,\omega)|^p\}<+\infty$, if $d=2$;

(iii) $\rho^{\mathrm{D}}(\lambda) \leqslant C_1 E\{|V(0,\omega)-\lambda-\eta|\chi(V(0,\omega)<\lambda+\eta)\}/\sqrt{\eta}$, $\quad\forall\eta>0$ if $d=1$;

(b) if $I=\mathbb{Z}^d$ for any $\lambda \in Q$:

(i) $\rho^{\mathrm{D}}(\lambda) \leqslant C_d E\left\{\int_{\Lambda_0} |V(x,\omega)-\lambda|^{d/2}\chi(V(x,\omega)<\lambda)\,\mathrm{d}x\right\}$ $\quad$ if $d\geqslant 3$;

(ii) $\rho^{\mathrm{D}}(\lambda) \leqslant C_2 E\left\{\int_{\Lambda_0} |V(x,\omega)-\lambda-\eta|^p\chi(V(x,\omega)<\lambda+\eta)\,\mathrm{d}x\right\}\Big/\eta^{p-1}$, $\quad\forall\eta>0$,

$p>1$ as in assumption A, if $d=2$;

(iii) $\rho^{\mathrm{D}}(\lambda) \leqslant C_1 E\left\{\int_{\Lambda_0} |V(x,\omega)-\lambda-\eta|\chi(V(x,\omega)<\lambda+\eta)\,\mathrm{d}x\right\}\Big/\sqrt{\eta}$,

$\forall\eta>0$ if $d=1$,

where C_d, C_2 and C_1 are the constants appearing in proposition 3.1, Λ_0 is as in assumption A and $\chi(V(0,\omega)<\lambda)$ is the characteristic function of the event $\{V(0,\omega)<\lambda\}$.

Proof.

(a) Since $\rho^{\mathrm{D}}(\lambda)=\sup(|\Lambda|^{-1}E\rho^{\mathrm{D}}_{\Lambda}(\lambda,\omega),\ \Lambda\in\mathscr{F},\ |\Lambda|>0)$ the estimates (i), (ii), (iii) follow immediately from proposition 3.1 and the strict stationarity of the random field $V(x,\omega)$.

(b) is proved in an analogous way.

Remark 3.1. In Krengel (1982) theorem 3.1, and therefore theorem 3.2 and corollary 3.1, is extended to the more general case where the superadditive process F is defined on the class of all Borel subsets of $\mathbb{R}^d$ and the limit is taken over a sequence of convex sets $(\Lambda_r)_{r\in Q}$ satisfying the following regularity conditions.

(i) $\beta(\Lambda_r)\to+\infty$ as $r\to+\infty$, where $\beta(\Lambda_r)$ is the supremum of all $s>0$ for which there exists a sphere $S(x,s)=(y\in\mathbb{R}^d;\ |y-x|\leqslant s)$ contained in Λ_r.

(ii) There exists a constant $C>0$ and a sequence Λ_r' in $\mathscr{F}$ such that $\Lambda_r'\subset\Lambda_r$ and $|\Lambda_r'|<C|\Lambda_r|$.

In this case one gets again the almost sure convergence of $\{F_{\Lambda_r}\}/|\Lambda_r|$.

Remark 3.2. The existence of the limit function $\rho^{\mathrm{D}}(\lambda)$, usually called the integrated density of states, was proved by Pastur (1973, 1971, 1972), and subsequently by Nakao (1977) (see also Fukushima (1974) and Fukushima *et al* (1975) for the case where the Laplacian is replaced by its discrete version on $l^2(\mathbb{Z}^d)$, Kotani (1976) and Nakao (1977) for the case where the potential V is a random measure on $\mathbb{R}$ and Gusev (1977) for an extension to more general elliptic operators) by very different methods, under the stronger assumptions that $V(x,\omega)$ is an ergodic random field on $\mathbb{R}^d$ with almost surely continuous sample paths and such that $E\{\exp[-tV(0,\omega)]\}<+\infty\ \forall t>0$.

It is an interesting problem to check if the non-decreasing function $\rho^{\mathrm{D}}(\lambda)$ actually depends on the boundary conditions (Dirichlet in our case) imposed at the boundary of the sets $\{\Lambda_r\}$. The next theorem shows that under an additional assumption on the negative part of the random field $V(x, \omega)$ the effect of the boundary conditions vanishes in the thermodynamic limit $\Lambda_r \to \mathbb{R}^d$.

Theorem 3.3. Let $V(x, \omega)$ satisfy assumption A and assume in addition that:

(a) if the index set I is $\mathbb{R}^d$ (i) $\sup \rho^{\mathrm{N}}_{[a,b)}(\lambda, \omega) \in L^1(\Omega, P)$ where the sup is taken over all the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $|a_i| \leqslant 1$, $|b_i| \leqslant 1$, a_i, $b_i \in Q$, $i = 1, \ldots, d$; (ii) $E\{\mathrm{Tr} \exp[-t_0(-\Delta^{\mathrm{N}}_{\Lambda_0} + qV)]\}$ is finite for some $t_0 > 0$ and some $q > 1$ where $-\Delta^{\mathrm{N}}_{\Lambda_0}$ is the Neumann Laplacian in $L^2(\Lambda_0)$, Λ_0 being as in assumption A;

(b) if $I = \mathbb{Z}^d$ only the second of the above conditions is necessary.

Then there exists on $\mathbb{R}$ a non-decreasing function $\rho^{\mathrm{N}}(\lambda)$ such that for any $\lambda \in Q$

$$\lim_{r \to +\infty} \frac{1}{|\Lambda_r|} \rho^{\mathrm{N}}_{\Lambda_r}(\lambda, \omega) = \rho^{\mathrm{N}}(\lambda) \qquad \text{AS}$$

where $\{\Lambda_r\}$ is a regular family in $\mathscr{F}$ or in $\mathscr{F}_1$ increasing to $\mathbb{R}^d$ and furthermore $\rho^{\mathrm{N}}(\lambda) = \rho^{\mathrm{D}}(\lambda)$ for almost all $\lambda \in \mathbb{R}$.

Proof. As usual we discuss only the case $I = \mathbb{R}^d$. We remark first of all that using proposition 2.2, for $\lambda \in Q$ fixed, the process $F_\Lambda(\omega) = -\rho^{\mathrm{N}}_\Lambda(\lambda, \omega)$ is superadditive in the sense of Akcoglu and Krengel and satisfies, by assumption, the additional condition of theorem 3.1. It follows that $\lim_{r \to +\infty} |\Lambda_r|^{-1} \rho^{\mathrm{N}}_{\Lambda_r}(\lambda, \omega)$, where (Λ_r) is as in the statement, exists almost surely and it is equal to $\inf(E\rho^{\mathrm{N}}_\Lambda(\lambda, \omega)/|\Lambda|, \Lambda \in \mathscr{F}, |\Lambda| > 0) \equiv \rho^{\mathrm{N}}(\lambda)$; the extension of $\rho^{\mathrm{N}}(\lambda)$ for $\lambda \in \mathbb{R} \backslash Q$ can be done as in theorem 3.2. Let us now estimate the difference $\rho^{\mathrm{N}}(\lambda) - \rho^{\mathrm{D}}(\lambda) \equiv G(\lambda)$. Since by proposition 2.2 $\rho^{\mathrm{N}}_\Lambda(\lambda, \omega) - \rho^{\mathrm{D}}_\Lambda(\lambda, \omega) \geqslant 0$, $\forall \Lambda \subset \mathbb{R}^d$, $\forall \lambda \in \mathbb{R}$ for almost all ω, we have that $G(\lambda) \geqslant 0$; on the other hand, using theorem 3.2 and the above result for $\rho^{\mathrm{N}}(\lambda)$, $G(\lambda) < |\Lambda|^{-1} E[\rho^{\mathrm{N}}_\Lambda(\lambda, \omega) - \rho^{\mathrm{D}}_\Lambda(\lambda, \omega)]$, $\forall \Lambda \subset \mathbb{R}^d$. It is then sufficient to show that $\lim_{r \to +\infty} |\Lambda_r|^{-1} E[\rho^{\mathrm{N}}_{\Lambda_r}(\lambda, \omega) - \rho^{\mathrm{D}}_{\Lambda_r}(\lambda, \omega)] = 0$ for a sequence of cubes Λ_r increasing to $\mathbb{R}^d$ or, using the estimate

$$\int_{-\infty}^{\lambda} (\rho^{\mathrm{N}}_\Lambda(\lambda', \omega) - \rho^{\mathrm{D}}_\Lambda(\lambda', \omega))\, \mathrm{d}\lambda' \leqslant \frac{1}{t} \mathrm{e}^{+t\lambda} (L^{\mathrm{N}}_\Lambda(t, \omega) - L^{\mathrm{D}}_\Lambda(t, \omega)), \qquad t > 0,$$

where

$$L^{\mathrm{N}}_\Lambda(t, \omega) \equiv \int_{-\infty}^{+\infty} \mathrm{e}^{-t\lambda}\, \mathrm{d}\rho^{\mathrm{N}}_\Lambda(\lambda, \omega) = \mathrm{Tr}\{\exp[-t(-\Delta^{\mathrm{N}}_\Lambda + V)]\}$$

and the same for $L^{\mathrm{D}}(t, \omega)$, that

$$\lim_{r \to +\infty} \frac{1}{|\Lambda_r|} E[L^{\mathrm{N}}_{\Lambda_r}(t, \omega) - L^{\mathrm{D}}_{\Lambda_r}(t, \omega)] = 0$$

for some $t > 0$. To prove this we need the following lemma.

Lemma 3.1. Let Λ be a hypercube of size M centred at $x = 0$ and let $V \in L^p(\Lambda)$ where p is as in lemma 2.1. Then

$$\begin{aligned} 0 &\leqslant \mathrm{Tr}\{\exp[-t(-\Delta^{\mathrm{N}}_\Lambda + V)]\} - \mathrm{Tr}\{\exp[-t(-\Delta^{\mathrm{D}}_\Lambda + V)]\} \\ &\leqslant \mathrm{Tr}\{\exp[-t(-\Delta^{\mathrm{N}}_\Lambda + qV)]\}^{1/q} \{\mathrm{Tr}[\exp(+t\Delta^{\mathrm{N}}_\Lambda)] - \mathrm{Tr}[\exp(t\Delta^{\mathrm{D}}_\Lambda)]\}^{1/s} \end{aligned}$$

$\forall q > 1$, $s > 1$ such that $q^{-1} + s^{-1} = 1$.

Proof. We remark first of all that using proposition 2.1 and a limiting argument it is enough to prove the lemma only for $V \in C_0^\infty(\Lambda)$. Now let $K_V^{\mathrm{N}}(x, y, t)$ and $K_V^{\mathrm{D}}(x, y, t)$ be the kernels of $\exp[-t(-\Delta_\Lambda^{\mathrm{N}} + V)]$ and $\exp[-t(-\Delta_\Lambda^{\mathrm{D}} + V)]$ respectively. It can be proved (see e.g. Bratteli and Robinson 1981) that for both K_V^{N} and K_V^{D} one has the following representation in terms of the Wiener measure:

$$K_V^{\mathrm{D}}(x, y, t) = \int \mathrm{d}P_{x,y}^t(\omega) \exp\left(-\int_0^t V(\omega_s)\,\mathrm{d}s\right)\chi(\omega_s \in \Lambda\ \forall s \leqslant t),$$

$$K_V^{\mathrm{N}}(x, y, t) = \sum_{u \in \varepsilon^{-1}(y)} \int \mathrm{d}P_{x,u}^t(\omega) \exp\left(-\int_0^t V[\varepsilon(\omega_s)]\,\mathrm{d}s\right),$$

where $P_{x,y}^t$ is the conditional Wiener measure on the space of continuous functions $\omega: [0, t] \to \mathbb{R}^d$ (see e.g. Simon 1979 and Nakao 1977) and ε is the function from $\mathbb{R}^d$ onto Λ defined by

$$\varepsilon(x)_i = x_i - kM \qquad \text{if } (2k - \tfrac{1}{2})M \leqslant x_i \leqslant (2k + \tfrac{1}{2})M,$$

$$\varepsilon(x)_i = (2k+1)M - x_i \qquad \text{if } (2k + \tfrac{1}{2})M \leqslant x_i \leqslant (2k + \tfrac{3}{2})M,$$

$i = 1, \ldots, d.$

Using the above representation we obtain immediately that

$$K_V^{\mathrm{N}}(x, x, t) - K_V^{\mathrm{D}}(x, x, t) = \sum_{u \in \varepsilon^{-1}(x)} \int \mathrm{d}P_{x,u}^t(\omega) \exp\left(-\int_0^t V[\varepsilon(\omega_s)]\,\mathrm{d}s\right)\chi_\Lambda(\omega)$$

where $\chi_\Lambda(\omega)$ is one if ω is such that $\omega_s \notin \Lambda$ for some $s \leqslant t$ and zero otherwise. Using now the Hölder inequality with exponents q and s, once in the Wiener integral and once in the sum, we obtain

$$K_V^{\mathrm{N}}(x, x, t) - K_V^{\mathrm{D}}(x, x, t) \leqslant \left[\sum_{u \in \varepsilon^{-1}(x)} \int \mathrm{d}P_{x,u}^t(\omega) \exp\left(-\int_0^t qV[\varepsilon(\omega_s)]\,\mathrm{d}s\right)\right]^{1/q},$$

$$\left(\sum_{u \in \varepsilon^{-1}(x)} \int \mathrm{d}P_{x,u}^t(\omega)\,\chi_\Lambda\right)^{1/s} = [K_{qV}^{\mathrm{N}}(x, xt)]^{1/q}[K_0^{\mathrm{N}}(x, x, t) - K_0^{\mathrm{D}}(x, x, t)]^{1/s},$$

where K_0^{N} and K_0^{D} are the kernels of $\exp(t\Delta_\Lambda^{\mathrm{N}})$ and $\exp(t\Delta_\Lambda^{\mathrm{D}})$ respectively.

Since for $V \in C_0^\infty(\Lambda)$ both $K_V^{\mathrm{N}}(x, y, t)$ and $K_V^{\mathrm{D}}(x, y, t)$ are jointly continuous in x and y for $t > 0$, we have that

$$\begin{aligned}
&\mathrm{Tr}\{\exp[-t(-\Delta_\Lambda^{\mathrm{N}} + V)]\} - \mathrm{Tr}\{\exp[-t(-\Delta_\Lambda^{\mathrm{D}} + V)]\}\\
&\qquad = \int_\Lambda \mathrm{d}x\,(K_V^{\mathrm{N}}(x, x, t) - K_V^{\mathrm{D}}(x, x, t))\\
&\qquad \leqslant [\![\mathrm{Tr}\{\exp[-t(-\Delta_\Lambda^{\mathrm{N}} + qV)]\}]\!]^{1/q}\{\mathrm{Tr}[\exp(t\Delta_\Lambda^{\mathrm{N}})] - \mathrm{Tr}[\exp(t\Delta_\Lambda^{\mathrm{D}})]\}^{1/s}
\end{aligned}$$

by the previous estimate on the difference between K_V^{N} and K_V^{D} and the Hölder inequality with exponents q and s applied to the integral over Λ.

We can now complete the proof of the theorem. Using the above lemma and again the Hölder inequality, we obtain

$$\begin{aligned}
&\frac{1}{|\Lambda_r|}E[L_{\Lambda_r}^{\mathrm{N}}(t, \omega) - L_{\Lambda_r}^{\mathrm{D}}(t, \omega)]\\
&\qquad \leqslant \{E[L^{\mathrm{N}}(t, \omega, q)]\}^{1/q}\left(\frac{1}{|\Lambda_r|}\{\mathrm{Tr}[\exp(t\Delta_{\Lambda_r}^{\mathrm{N}})] - \mathrm{Tr}[\exp(t\Delta_{\Lambda_r}^{\mathrm{D}})]\}\right)^{1/s}
\end{aligned}$$

where $L^{N}_{\Lambda_r}(t, \omega, q)$ is the same as $L^{N}_{\Lambda_r}(t, \omega)$ but computed for $-\Delta^{N}_{\Lambda_r} + qV(\cdot, \omega)$. Assume now that Λ_r has size $r \in N$ and let $\{\tilde{\Lambda}_i\}$ be the collection of the basic cells of $\mathbb{Z}^d$ inside Λ_r. Since $-\Delta^{N}_{\Lambda_r} + V(\cdot, \omega)_- \leq \bigoplus_i (-\Delta^{N}_{\tilde{\Lambda}_i} + V(\cdot, \omega))$ where $-\Delta^{N}_{\tilde{\Lambda}_i}$ is the Neumann Laplacian on $L^2(\tilde{\Lambda}_i)$ and $\bigoplus_i$ is the direct sum (see e.g. Reed and Simon 1978b), we obtain:

$$\frac{1}{|\Lambda_r|} E[L^{N}_{\Lambda_r}(t, \omega, q)] \leq \frac{1}{|\Lambda_r|} E\left(\sum_i L^{N}_{\tilde{\Lambda}_i}(t, \omega, q)\right) = E[L^{N}_{\Lambda_0}(t, \omega, q)] < +\infty$$

by assumption; since $|\Lambda_r|^{-1}\{\mathrm{Tr}[\exp(t\Delta^{N}_{\Lambda_r})] - \mathrm{Tr}[\exp(t\Delta^{D}_{\Lambda_r})]\} \to 0$ as $r \to +\infty$ (see e.g. Pastur 1971) the theorem is proved.

Remark 3.3. Benderskii and Pastur (1970) proved the above result with only the assumption of the existence of the first moment of $|V(x, \omega)|$ in the one-dimensional case, using methods from the Sturm–Liouville theory. For a more detailed analysis of $\rho(\lambda)$ in the one-dimensional case we refer to the paper by Molchanov (1981). We also notice that the theorem covers the case of mixed and periodic boundary conditions since they are in between the Dirichlet and Neumann case.

We end the section by giving a sufficient condition on the random field $V(x, \omega)$ such that the conditions of the previous theorem in the case when the index set I is $\mathbb{R}^d$ are satisfied.

Theorem 3.4. Assume $E\{\exp[-tV(0, \omega)]\} < +\infty$ for some $t > 0$ (and hence for all $s < t$). Then

(i) $\sup \rho^{N}_{\Lambda}(\lambda, \omega) \in L^1(\Omega, P)$ where the sup is taken over all cubes inside Λ_0;

(ii) $E\{\mathrm{Tr}[\exp(-\Delta^{N}_{\Lambda} + V_\omega)]\} \leq E\{\exp[-tV(0, \omega)]\, \mathrm{Tr}[\exp(t\Delta^{N}_{\Lambda})]\}$ for any bounded hypercube Λ in $\mathbb{R}^d$;

(iii) let $\rho(\lambda) = \rho^{N}(\lambda) = \rho^{D}(\lambda)$ according to theorem 3.3 and let $L(t)$ be its Laplace transform; then

$$L(t) \leq (2\pi t)^{-d/2} E\{\exp[-tV(0, \omega)]\}.$$

Proof.

(i) Let $V^-(x, \omega)$ denote the negative part of $V(x, \omega)$. Then, using the assumption $E\{\exp[-tV(0, \omega)]\} < +\infty$, it is easy to show that $E\{(\int_\Lambda |V^-(x, \omega)|^p \, dx)^m\} < +\infty$ for any bounded hypercube $\Lambda \subset \mathbb{R}^d$, $\forall p$, $m < +\infty$. Let now Λ be fixed and let, for $1 > \varepsilon > 0$, $b_\Lambda(\varepsilon, V^-)$ be the constant appearing in lemma 2.1 computed for $V^-(x, \omega)$. It is not difficult to check, using the Sobolev embedding theorems, that $b_\Lambda(\varepsilon, V^-)$ can be taken proportional to $(\int_\Lambda |V^-(x, \omega)|^p \, dx)^m$ for some p and m sufficiently large, depending only on the dimension d. Using now the estimate of proposition 2.1 on the eigenvalues of $H^{N}_{\Lambda}(\omega)$, we obtain

$$\begin{aligned} \rho^{N}_{\Lambda}(\lambda, \omega) &\leq \#\left\{k \in \mathbb{N};\ \mu^{N}_{k} \leq \frac{\lambda + b_\Lambda(\varepsilon, V^-_\omega)}{(1-\varepsilon)}\right\} \\ &\leq \text{constant}\, |\Lambda| \left|\frac{\lambda + b_\Lambda(\varepsilon, V^-)}{1-\varepsilon}\right|^{d/2} \\ &\quad + \text{constant}\left(1 + \text{constant}\, |\Lambda| \left|\frac{\lambda + b_\Lambda(\varepsilon, V^-)}{(1-\varepsilon)}\right|^{(d-1)/2}\right) \end{aligned}$$

(see e.g. Reed and Simon 1978b, ch XIII, 15, proposition 2). It is clear that the supremum over Λ in the basic cell Λ_0 of this last expression is in $L^1(\Omega, P)$.

(ii) Let $\Lambda \subset \mathbb{R}^d$ be fixed and let for $M > 0$ $V_M(x, \omega) = \max(-M, V(x, \omega))$. Then, by the Golden–Thompson inequality (see e.g. Reed and Simon 1978b) we have

$$\operatorname{Tr}\{\exp[-t(-\Delta_\Lambda^N + V_M)]\} \leqslant \operatorname{Tr}[\exp(\tfrac{1}{2}t\Delta_\Lambda^N)\exp(-tV_M)\exp(\tfrac{1}{2}t\Delta_\Lambda^N)]$$
$$= \sum_{n=1}^{\infty} \exp(-t\mu_n^N)\int_\Lambda |\varphi_n(x)|^2 \exp[-tV_M(x, \omega)]\,dx,$$

where $\{\varphi_n\}$ and $\{\mu_n^N\}$ are the eigenfunctions and eigenvalues of the Neumann Laplacian on $L^2(\Lambda)$. Taking now the expectation of both sides of this inequality, using the strict stationarity of $V(x, \omega)$ and the obvious estimate $E\{\exp[-tV_M(x, \omega)]\} \leqslant E\{\exp[-tV(x, \omega)]\}$, we obtain

$$E[\![\operatorname{Tr}\{\exp[-t(-\Delta_\Lambda^N + V_M(\cdot, \omega))]\}]\!] \leqslant \operatorname{Tr}[\exp(t\Delta_\Lambda^N)]E\{\exp[-tV(0, \omega)]\}$$

for any $M > 0$. Since as $M \to +\infty$, $V_M \to V$ in $L^p(\Lambda)$ where p is as in lemma 2.1, we obtain the statement by taking the limit $M \to +\infty$ in the last inequality, using the result (e) of proposition 2.1 and the monotone convergence theorem.

(iii) Let $\rho(\lambda) = \rho^N(\lambda) = \rho^D(\lambda)$; as we have already remarked $\rho^N(\lambda) \leqslant |\Lambda|^{-1}E[\rho_\Lambda^N(\lambda, \omega)]$ for any hypercube Λ in $\mathbb{R}^d$, so that

$$\int_{-\infty}^{+\infty} \exp(-t\lambda)\,d\rho^N(\lambda) \leqslant \frac{1}{|\Lambda|}\int_{-\infty}^{+\infty} \exp(-t\lambda)\,d\{E[\rho_\Lambda^N(\lambda, \omega)]\}$$
$$= \frac{1}{|\Lambda|}E\int_{-\infty}^{+\infty} \exp(-t\lambda)\,d\rho_\Lambda^N(\lambda, \omega),$$

by Fubini's theorem; this last expression using (ii) is bounded by

$$|\Lambda|^{-1}E\{\exp[-tV(0, \omega)]\}\operatorname{Tr}[\exp(t\Delta_\Lambda^N)]$$

so that taking the limit $\Lambda \to \mathbb{R}^d$ and observing that $|\Lambda|^{-1}\operatorname{Tr}[\exp(t\Delta_\Lambda^N)]$ converges as $\Lambda \to \mathbb{R}^d$ to $(2\pi t)^{-d/2}$ (see e.g. Pastur 1971) we get the result.

Remark 3.4. Estimate (iii) on $L(t)$ was proved under the stronger assumption of the continuity of the sample paths of $V(x, \omega)$ by Pastur (1977), Fukushima (1974) and Nakao (1977) using a representation of $L(t)$ as a Wiener integral. In the discrete case $I = \mathbb{Z}^d$ the situation is more complicated since the random field $V(x, \omega)$ is no longer strictly stationary. However in order to assure the boundedness of $E\{\operatorname{Tr}[\exp(-t(-\Delta_{\Lambda_0}^N + V(\cdot, \omega)))]\}$ we can use lemma 2.1 and obtain

$$E\{\operatorname{Tr}[\exp(-t(-\Delta_{\Lambda_0}^N + V(\cdot, \omega)))]\} \leqslant E\{\exp[b_0(\varepsilon, V_\omega)t]\}\operatorname{Tr}[\exp(\varepsilon t\Delta_{\Lambda_0}^N)]$$

where $b_0(\varepsilon, V)$ is the constant appearing in lemma 2.1 computed for V_ω restricted to Λ_0; as has already been remarked, using the Sobolev embedding theorems, $b_0(\varepsilon, V_\omega)$ can be taken proportional to $\|V_\omega\|_{L^q(\Lambda_0)}^k$ for sufficiently large k and q, provided $\|V_\omega\|_{L^q(\Lambda_0)}^k < +\infty$ almost surely. Here ε is an arbitrary number between 0 and 1. We also remark that results analogous to the above theorem can be obtained by combining the path integral approach to the density of states used by Pastur and recent estimates obtained by Carmona (1979) on Wiener integrals.

4. Asymptotic results for $\lambda \to \pm\infty$

We prove here some asymptotic results for the limit function $\rho^D(\lambda)$ in the case when λ goes to plus or minus infinity. They can be looked upon as an extension to our case of previous results obtained by Pastur (1973) and Nakao (1977). Throughout the section the random field $V(x, \omega)$ is assumed to satisfy assumption A and $\rho^{D}(\lambda)$ will denote the corresponding limit function constructed according to theorem 3.2.

Theorem 4.1.

$$\lim_{\lambda \to +\infty} \rho^{D}(\lambda)\lambda^{-d/2} = (2\pi)^{-d}\tau_d$$

where τ_d is the volume of the unit sphere in $\mathbb{R}^d$.

Proof. As before we give the proof only in the case when the index set I of assumption A is $\mathbb{R}^d$, the other case being analogous. We remark first of all that, using corollary 3.1(a),

$$\lim_{\lambda \to +\infty} \lambda^{-d/2}\rho^{D}(\lambda) < +\infty \qquad \forall d \geqslant 1.$$

For $d \geqslant 3$ this is obvious from the estimate $\rho^{D}(\lambda) \leqslant C_d E(|V(0, \omega) - \lambda|^{d/2})$. For $d = 2$ or $d = 1$ the same holds if we take the arbitrary constant $\eta > 0$ appearing in estimates (ii) and (iii) of corollary 3.1(a) proportional to λ. Let us now prove the correct lower bound for the limit. Since $\rho^{D}(\lambda) = \sup_{\Lambda \in \mathscr{F}} |\Lambda|^{-1} E[\rho_{\Lambda}^{D}(\lambda, \omega)]$ we have

$$\varliminf_{\lambda \to +\infty} \lambda^{-d/2}\rho^{D}(\lambda) \geqslant \varliminf_{\lambda \to +\infty} \lambda^{-d/2}\frac{E}{|\Lambda|}[\rho_{\Lambda}^{D}(\lambda, \omega)] \qquad \forall \Lambda \subset \mathbb{R}^d.$$

Using now Fatou's lemma and the result of proposition (2.3), we obtain

$$\varliminf_{\lambda \to +\infty} \lambda^{-d/2}\frac{E}{|\Lambda|}[\rho^{D}(\lambda, \omega)] \geqslant (2\pi)^{-d}\tau_d.$$

Upper bound. We first discuss the case $d \geqslant 3$. Let $\Lambda \subset \mathbb{R}^d$ be fixed and let us write $E[\rho_{\Lambda}^{D}(\lambda, \omega)]/|\Lambda|$ as

$$|\Lambda|^{-1}E[\rho_{\Lambda}^{D}(\lambda, \omega)] = |\Lambda|^{-1}E[\#(n \in N; \lambda_n[H_{\Lambda}^{D}(\omega) - \lambda] < 0)]. \tag{5}$$

We now decompose the random field $V(x, \omega) - \lambda$ in two parts:

$$W_1(x, \lambda, \omega) = (V(x, \omega) - \lambda)\chi(V(x, \omega) \leqslant \lambda(1 - \alpha)),$$
$$W_2(x, \lambda, \omega) = (V(x, \omega) - \lambda)\chi(V(x, \omega) > \lambda(1 - \alpha)),$$

where $\alpha > 1$ is an arbitrary constant, so that, for any $1 > \varepsilon > 0$, the operator $H_{\Lambda}^{D}(\omega) - \lambda$ can be written as

$$H_{\Lambda}^{D}(\omega) - \lambda = H_{\Lambda,1}^{D,\varepsilon}(\omega) + H_{\Lambda,2}^{D,\varepsilon}(\omega)$$

where

$$H_{\Lambda,1}^{D,\varepsilon}(\omega) = -(1 - \varepsilon)\Delta_{\Lambda}^{D} + W_1(\omega) \qquad \text{and} \qquad H_{\Lambda,2}^{D,\varepsilon}(\omega) = -\varepsilon\Delta_{\Lambda}^{D} + W_2(\omega).$$

Using this decomposition, (5) is bounded by

$$|\Lambda|^{-1}E[\#(n \in N; \lambda_n[H_{\Lambda,1}^{D,\varepsilon}(\omega)] < 0)] + |\Lambda|^{-1}E[\#(n \in N; \lambda_n[H_{\Lambda,2}^{D,\varepsilon}(\omega)] < 0)] \tag{6}$$

(see e.g. Reed and Simon 1978b, theorem XIII 80).

We now estimate separately each term of (6). The first term, using proposition 3.1 and the stationarity of $V(x, \omega)$, is bounded by

$$\frac{1}{|\Lambda|}E\left\{\int_{\Lambda}\left|\frac{W_1(x,\omega,\lambda)}{\varepsilon}\right|^{d/2}\mathrm{d}x\right\}=E\left\{\left|\frac{V(0,\omega)-\lambda}{\varepsilon}\right|^{d/2}\chi(V(0,\omega)\leqslant\lambda(1-\alpha))\right\}. \tag{7}$$

The second term, using the bound $W(x, \omega, \lambda)\geqslant-\alpha\lambda$, is estimated by

$$\frac{1}{|\Lambda|}E\left\{\#\left(n\in\mathbb{N};\,\lambda_n(-\Delta_\Lambda^{\mathrm{D}})\leqslant\frac{\alpha\lambda}{1-\varepsilon}\right)\right\}$$
$$\leqslant(2\pi)^{-d}\tau_d\left|\frac{\alpha\lambda}{1-\varepsilon}\right|^{d/2}+\text{constant}\left(|\Lambda|^{-1}+|\Lambda|^{-1/d}\left|\frac{\alpha\lambda}{1-\varepsilon}\right|^{(d-1)/2}\right) \tag{8}$$

(see e.g. Reed and Simon 1978b, ch XIII 15, proposition 2 for a proof of this last inequality). Combining now (5), (6), (7) and (8) we obtain

$$\frac{1}{|\Lambda|}E[\rho_\Lambda^{\mathrm{D}}(\lambda,\omega)]\leqslant E\left\{\left|\frac{V(0,\omega)-\lambda}{\varepsilon}\right|^{d/2}\chi(V(0,\omega)<\lambda(1-\alpha))\right\}$$
$$+(2\pi)^{-d}\tau_d\left|\frac{\alpha\lambda}{1-\varepsilon}\right|^{d/2}+\text{constant}\left(|\Lambda|^{-1}+|\Lambda|^{-1/d}\left|\frac{\alpha\lambda}{1-\varepsilon}\right|^{(d-1)/2}\right),$$
$$\forall\varepsilon>0,\forall\alpha>1. \tag{9}$$

Taking now in (9) the limit $\Lambda_n\to\mathbb{R}^d$, where $\{\Lambda_n\}$ is a regular family in $\mathscr{F}$ increasing to $\mathbb{R}^d$, we obtain using theorem 3.2

$$\rho^{\mathrm{D}}(\lambda)\leqslant E\left\{\left|\frac{V(0,\omega)-\lambda}{\varepsilon}\right|^{d/2}\chi(V(0,\omega)\leqslant\lambda(1-\alpha))\right\}$$
$$+(2\pi)^{-d}\tau_d\left|\frac{\alpha\lambda}{1-\varepsilon}\right|^{d/2},\qquad\forall\varepsilon>0,\forall\alpha>1.$$

Dividing by $\lambda^{-d/2}$ and passing first to the limit $\lambda\to+\infty$ and subsequently to the limits $\varepsilon\to0$, $\alpha\to1$, we obtain

$$\overline{\lim_{\lambda\to+\infty}}\,\lambda^{-d/2}\rho^{\mathrm{D}}(\lambda)\leqslant(2\pi)^{-d}\tau_d.$$

The proof in the case $d=2$ or $d=1$ goes along the same lines with the only difference that estimate (7) must be replaced by

$$E\{|(V(0,\omega)-\lambda)\chi(V(0,\omega)<\lambda(1-\alpha))-\eta|^p\}/\eta^{p-1},\qquad\forall\eta>0,p>1,$$

as in assumption A, if $d=2$, and by

$$E\{|(V(0,\omega)-\lambda)\chi(V(0,\omega)<\lambda(1-\alpha))-\eta|\}/\sqrt{\eta},\qquad\forall\eta>0\text{ if }d=1.$$

It is then sufficient to take the arbitrary constant $\eta=\sigma\lambda$, $\sigma>0$, and pass to the limit $\sigma\to0$ after the limits $\lambda\to+\infty$, $\varepsilon\to0$, $\alpha\to1$.

We now turn to examine the behaviour of $\rho^{\mathrm{D}}(\lambda)$ as $\lambda\to-\infty$.

Theorem 4.2.

(i) Let $\gamma>0$ be such that $\gamma+d/2>1$ and assume that $E\{|V^-(0,\omega)|^{\gamma+d/2}\}<+\infty$ in the case when the index set I is $\mathbb{R}^d$, or $E\{\int_{\Lambda_0}|V^-(x,\omega)|^{\gamma+d/2}\,\mathrm{d}x\}<+\infty$ in the case

$I = \mathbb{Z}^d$. Then

$$\int_{-\infty}^{-1} |\lambda|^\gamma \, \mathrm{d}\rho^{\mathrm{D}}(\lambda) \leqslant K_{d,\gamma} E\{|V(0,\omega)|^{\gamma+d/2}\} \qquad \text{if } I = \mathbb{R}^d,$$

$$\int_{-\infty}^{-1} |\lambda|^\gamma \, \mathrm{d}\rho^{\mathrm{D}}(\lambda) \leqslant K_{d,\gamma} E\left\{\int_{\Lambda_0} |V(x,\omega)|^{\gamma+d/2} \, \mathrm{d}x\right\} \qquad \text{if } I = \mathbb{Z}^d.$$

Here as usual $V^-(x, \omega)$ denotes the negative part of $V(x, \omega)$ and $K_{d,\gamma}$ is a positive constant.

(ii) If the index set I is $\mathbb{R}^d$, $\lim_{\lambda \to -\infty} -|\lambda|^{-\alpha} \ln[P(V(0,\omega) < \lambda)] \leqslant B$, $\alpha > 0$, $B > 0$ implies

$$\lim_{\lambda \to -\infty} -|\lambda|^{-\alpha} \ln[\rho^{\mathrm{D}}(\lambda)] \leqslant B.$$

Proof.

(i) Since $\rho^{\mathrm{D}}(\lambda)$ as a measure on $\mathbb{R}$ is the weak limit of $\rho^{\mathrm{D}}_{\Lambda_n}(\lambda, \omega)/|\Lambda_n|$ for almost all ω, where $\{\Lambda_n\}$ is a regular family in $\mathcal{F}$ with $\Lambda_n \to \mathbb{R}^d$ (or $\Lambda_n \in \mathcal{F}_1$ in the case $I = \mathbb{Z}^d$), we have

$$\begin{aligned}
\int_{-\infty}^{-1} |\lambda|^\gamma \, \mathrm{d}\rho^{\mathrm{D}}(\lambda) &= \lim_{k \to -\infty} \int_k^{-1} |\lambda|^\gamma \, \mathrm{d}\rho^{\mathrm{D}}(\lambda) \\
&= \lim_{k \to -\infty} \lim_{n \to \infty} \frac{1}{|\Lambda_n|} \int_k^{-1} |\lambda|^\gamma \, \mathrm{d}\rho_{\Lambda_n}(\lambda, \omega) \\
&\leqslant \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} \int_{-\infty}^{-1} |\lambda|^\gamma \, \mathrm{d}\rho_{\Lambda_n}(\lambda, \omega) \\
&= \lim_{n \to +\infty} \frac{1}{|\Lambda_n|} \sum_i |\lambda_i[H^{\mathrm{D}}_{\Lambda_n}(\omega)]|^\gamma \chi(\lambda_i[H^{\mathrm{D}}_{\Lambda_n}(\omega)] < -1). \qquad (10)
\end{aligned}$$

Since $\lambda_i[H^{\mathrm{D}}_{\Lambda_n}(\omega)] \geqslant \lambda_i[-\Delta + V^-(\cdot, \omega)\chi_{\Lambda_n}] \, \forall i$, where $-\Delta + V^-(\cdot, \omega)\chi_{\Lambda_n}$ acts on $L^2(\mathbb{R}^d)$ and $\chi_{\Lambda_n}(x) = 1$ if $x \in \Lambda_n$, zero otherwise (see e.g. Reed and Simon 1978b), we get that (10) is bounded by

$$\begin{aligned}
&\lim_{n \to +\infty} \frac{1}{|\Lambda_n|} \sum_i |\lambda_i[-\Delta + V^-(\cdot, \omega)\chi_{\Lambda_n}]|^\gamma \chi(\lambda_i[-\Delta + V^-(\cdot, \omega)\chi_{\Lambda_n}] < 0) \\
&\qquad \leqslant \lim_{n \to +\infty} K_{d,\gamma} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} |V^-(x, \omega)|^{\gamma+d/2} \, \mathrm{d}x = K_{d,\gamma} E\{|V^-(0, \omega)|^{\gamma+d/2}\}
\end{aligned}$$

in the case $I = \mathbb{R}^d$, or equal to $E\{\int_{\Lambda_0} |V^-(x, \omega)|^{\gamma+d/2} \, \mathrm{d}x\}$ in the case $I = \mathbb{Z}^d$. Here we have used the Lieb–Thirring bound on the sum of the γ-power of the negative eigenvalues of $-\Delta + V$ on $L^2(\mathbb{R}^d)$ (see Lieb and Thirring 1976):

$$\sum_i |\lambda_i(-\Delta + V)|^\gamma \chi(\lambda_i(-\Delta + V) < 0) \leqslant K_{d,\gamma} \int_{\mathbb{R}^d} |V^-(x)|^{\gamma+d/2} \, \mathrm{d}x$$

where $K_{d,\gamma}$ is a positive constant depending only on d and γ, and the ergodic theorem for metrically transitive random fields (see e.g. Tempelmann 1972).

(ii) It is a straightforward consequence of the estimate (i), (ii) and (iii) of corollary 3.1(b).

We conclude with two examples. The first one shows that in general the estimates of corollary 3.1(b) and of the above theorem cannot be improved; the second one generalises a result on the asymptotics of $\rho^{\mathrm{D}}(\lambda)$ as $\lambda \to -\infty$ obtained by Pastur (1977) and Nakao (1977) for gaussian random fields $V(x, \omega)$ with continuous sample paths, to the case when this condition is not assumed.

Example 1. Let $\{q_i(\omega)\}_{i\in\mathbb{Z}^d}$ be a stationary metrically transitive random field on $\mathbb{Z}^d$ such that $E\{|q_0(\omega)|^p\}<+\infty$ where p is as in lemma 2.1, and let also $V(x, \omega)=q_i(\omega)$ $\forall x \in \Lambda_0+i$, Λ_0 being as in assumption A. Then $V(x, \omega)$, as a random field on $\mathbb{R}^d$, satisfies assumption A, and the associated limit function $\rho^{\mathrm{D}}(\lambda)$ constructed according to theorem 3.2 is estimated from above by

$$\rho^{\mathrm{D}}(\lambda) \leq C_d E\left\{\int_{\Lambda_0} |V(x, \omega)-\lambda|^{d/2}\chi(V(x, \omega)<\lambda)\,\mathrm{d}x\right\}$$

$$= C_d E\{|q_0(\omega)-\lambda|^{d/2}\chi(q_0(\omega)<\lambda)\} \tag{11}$$

in the case $d \geq 3$ and analogously for $d=2$ and $d=1$ (see corollary 3.1(b)). Let us now assume for example that for some $\alpha>p$ $\lim_{\lambda\to-\infty}|\lambda|^\alpha P(q_0(\omega)<\lambda)=B$, B being an arbitrary finite constant. As can be checked, (11) implies that

$$\lim_{\lambda\to-\infty} |\lambda|^{\alpha-d/2}\rho^{\mathrm{D}}(\lambda) \leq \text{constant } B.$$

On the other hand, according to theorem 3.2 $\rho^{\mathrm{D}}(\lambda)=\sup_{\Lambda\in\mathscr{F}_1} E|\Lambda|^{-1}\{\rho^{\mathrm{D}}_{\Lambda}(\lambda, \omega)\}$ so that

$$\begin{aligned}\rho^{\mathrm{D}}(\lambda) &\geq E\{\rho^{\mathrm{D}}_{\Lambda_0}(\lambda, \omega)\}=E\{\#(n\in\mathbb{N};\ \lambda_n(-\Delta^{\mathrm{D}}_{\Lambda_0})<\lambda-q_0(\omega)\}\\ &\geq E\{\tau_d(2\pi)^{-d}|q_0-\lambda|^{d/2}(1-\text{constant}(|q_0(\omega)-\lambda|^{-d/2}\\ &\quad+|q_0(\omega)-\lambda|^{-1/2}))\chi(q_0(\omega)<\sigma\lambda)\}, \qquad \forall\sigma>1,\end{aligned} \tag{12}$$

(see again Reed and Simon 1978b for this estimate). From (12) it follows immediately that

$$\lim_{\lambda\to-\infty} |\lambda|^{\alpha-d/2}\rho^{\mathrm{D}}(\lambda)>(2\pi)^{-d}B\tau_d\frac{(\sigma-1)^{d/2}}{\sigma^\alpha} \qquad \forall\sigma>1.$$

It is worthwhile to remark that in the case when the Laplacian $-\Delta$ is replaced by its discrete version on $l^2(\mathbb{Z}^d)$ it is possible to prove (see Fukushima 1980) that $\rho^{\mathrm{D}}(\lambda)$ goes to zero as λ goes to minus infinity precisely as the quantity $P(V(0, \omega)<\lambda)$ (actually $\rho^{\mathrm{D}}(\lambda)$ in the discrete case is independent of the boundary conditions under very general assumptions on the random field $V(x, \omega)$, $x\in\mathbb{Z}^d$); a crucial role in this result was played by the boundedness of the discrete Laplacian on $l^2(\mathbb{Z}^d)$. The above example and corollary 3.1(b) show that in the continuous case a correction proportional to $|\lambda|^{d/2}$ is present.

Example 2. Let $V(x, \omega)$ be a jointly measurable gaussian random field on $\mathbb{R}^d$ with correlation function $h(x-y)=E\{V(x, \omega)V(y, \omega)\}$. Without loss of generality we can also assume that $E\{V(0, \omega)\}=0$. Let $\rho^{\mathrm{D}}(\lambda)$ be the associated limit function; since the conditions of theorems 3.4 and 3.2 are clearly satisfied, $\rho^{\mathrm{D}}(\lambda)$ is actually independent of the boundary conditions, so we will write $\rho(\lambda)$ instead of $\rho^{\mathrm{D}}(\lambda)$. Then for $\rho(\lambda)$ one has the following asymptotic result:

$$\lim_{\lambda\to-\infty} -\lambda^{-2}\ln[\rho(\lambda)]=[2h(0)]^{-1}.$$

Proof. Using theorem 4.2 (ii) it is enough to prove the lower bound

$$\varliminf_{\lambda\to-\infty} -\lambda^{-2}\ln[\rho(\lambda)]\geqslant[2h(0)]^{-1}. \tag{13}$$

As in example 1 we can bound $\rho(\lambda)$ from below by $E\{\rho^{\mathrm{D}}_{\Lambda_0}(\lambda,\omega)\}$. Obviously

$$\begin{aligned}E\{\rho^{\mathrm{D}}_{\Lambda_0}(\lambda,\omega)\}&\geqslant P\{\lambda_1[-\Delta^{\mathrm{D}}_{\Lambda_0}+V(\cdot,\omega)]<\lambda\}\\&\geqslant P\left\{\int_{\Lambda_0}\mathrm{d}x\,|\varphi(x)|^2V(x,\omega)<\lambda-\int_{\Lambda_0}|\nabla\varphi(x)|^2\,\mathrm{d}x\right\}\ \forall\varphi\in H^1_0(\Lambda_0)\end{aligned} \tag{14}$$

with $L^2(\Lambda_0)$-norm equal to one, since by the min–max principle

$$\lambda_1[-\Delta^{\mathrm{D}}_{\Lambda_0}+V(\cdot,\omega)]=\inf_{\substack{\varphi\in H_0(\Lambda_0)\\ \int_{\Lambda_0}|\varphi(x)|^2\,\mathrm{d}x=1}}\left\{\int_{\Lambda_0}|\nabla\varphi(x)|^2\,\mathrm{d}x+\int_{\Lambda_0}|\varphi(x)|^2V(x,\omega)\,\mathrm{d}x\right\}.$$

Using now the gaussian character of $V(x,\omega)$, it is easy to show (by computing e.g. the characteristic function) that $\xi_\varphi\equiv\int_{\Lambda_0}\mathrm{d}x\,|\varphi(x)|^2V(x,\omega)$ is again a gaussian random variable with mean zero and variance $\sigma_\varphi=\int_{\Lambda_0}\mathrm{d}x\int_{\Lambda_0}\mathrm{d}y\,|\varphi(x)|^2|\varphi(y)|^2h(x-y)$, so that (14) implies

$$\varliminf_{\lambda\to-\infty} -\lambda^{-2}\ln[\rho(\lambda)]\geqslant[2\sigma_\varphi]^{-1},\qquad \forall\varphi\in H^1_0(\Lambda_0)\text{ with }\|\varphi\|_{L^2(\Lambda_0)}=1. \tag{15}$$

Finally we observe that since $h(x)$ is a measurable positive definite function on $\mathbb{R}^d$ satisfying the estimate $|h(x)|\leqslant h(0)$, it is also continuous (see e.g. Gelfand and Vilenkin 1966), so that taking in (15) a normalised δ-sequence φ_n, $\varphi_n\in H^1_0(\Lambda_0)\ \forall n$, shrinking at the point $x=0$ we obtain

$$\varliminf_{\lambda\to-\infty} -\lambda^{-2}\ln[\rho(\lambda)]\geqslant[2h(0)]^{-1}.$$

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